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THE TOPOLOGY OF CERTAIN RIEMANNIAN MANIFOLDS WITH POSITIVE RICCI CURVATURE

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1. Let M be a complete connected Riemannian manifold of dimension n, and let Ric denote its Ricci curvature. Understanding the Ricci curvature is one of the important problems in today's geometry. In these notes, we assume that $\mathrm{Ric} \geqslant n-1$. The classical theorem of Myers then asserts that M is compact and has diameter $d_M \leqslant \pi$. R. Bishop showed that the volume of M also satisfied $\mathrm{vol}_M \leqslant \mathrm{vol}_{S^n}$, where S^n is the unit Euclidean sphere in $\mathbf{R}^{n+1}$, and that the equality holds only if M is isometric to S^n. In [3], S. Y. Cheng proves

Theorem A. *If $d_M = \pi$, then M is isometric to S^n.*

It is interesting to ask to what extent these theorems can be perturbed. Our main result is

Main Theorem. *Given any upper bound κ for the sectional curvature of M, there exists a constant $v > 0$, depending only on n and κ, such that whenever $\mathrm{vol}_M \geqslant (1-v)\mathrm{vol}_{S^n}$, then M has the homotopy type of S^n.*

By using some of the same methods, we can also show

Theorem B. *There is a constant $\rho > 0$, depending only on n, such that if M has the injectivity radius $i_M > \pi - \rho$, then M is homeomorphic to S^n.*

In §2 of these notes, we describe the main tools which can be used to prove these theorems. In §§3 and 4, we outline the proofs of Theorem B and Main Theorem. In §5, we describe a new geometric proof for Theorem A. Finally, we discuss some remarks and open question in §6. Details and additional applications will appear in [10]. The author would like to express gratitude to D. Gromoll for many helpful discussions.

2. Our main tool is the following observation in [7], based on an earlier work by Bishop. We denote by $B(r; p)$ the open metric ball of radius r and center p in M, and let $\hat{B}(r)$ be an open ball in S^n of radius r. Then we have

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Lemma 2.1 (*M. Gromov*). *For any $R \geqslant r > 0$,*

$$\frac{\operatorname{vol}_{B(R;\,p)}}{\operatorname{vol}_{B(r;\,p)}} \leqslant \frac{\operatorname{vol}_{\hat{B}(R)}}{\operatorname{vol}_{\hat{B}(r)}}.$$

Putting $R = \pi$ and cross-multiplying, we obtain

Lemma 2.2. *If for some $0 < v < 1$, $\operatorname{vol}_M > (1 - v)\operatorname{vol}_{S^n}$, then for any $r > 0$ and $p \in M$, we have*

$$\operatorname{vol}_{B(r;\,p)} > (1 - v)\operatorname{vol}_{\hat{B}(r)}.$$

It also follows from Lemma 2.1 that if $\operatorname{vol}_M > (1 - v)\operatorname{vol}_{S^n}$ for some v, then d_M must exceed the radius of the ball in S^n of volume $(1 - v)\operatorname{vol}_{S^n}$. We call this radius $D(v)$.

Lemma 2.3. *Let $\operatorname{vol}_M > (1 - v)\operatorname{vol}_{S^n}$. Let $p, q \in M$ have distance $d(p, q) = d_M$. Then given any $0 < d_1 \leqslant d_2$ with $d_1 + d_2 = D(v)$, there is an $r > 0$ such that the closed balls $B(d_1 + r;\, p)^-$ and $B(d_2 + r;\, q)^-$ cover M. Moreover, for fixed d_1, r can be so chosen as to go to 0 as v approaches 0.*

For the proof of the above, we estimate the volume of the complement of the set $B(d_1;\, p) \cup B(d_2;\, q)$ and the volume of $B(r;\, x)$ at an arbitrary x in this complement. If the latter exceeds the former, $B(r;\, x)^-$ must intersect $B(d_1;\, p)^- \cup B(d_2;\, q)^-$; consequently either $x \in B(d_1 + r;\, p)^-$ or $x \in B(d_2 + r;\, q)^-$.

3. We now outline the proof of Theorem B. Setting $d_1 = d_2$ in Lemma 2.3, the next proposition can be obtained by a construction similar to that in the proof of the classical Sphere Theorem (cf. [2, Chap. 6] or [6, §7.8]). In the following, $i(p)$ is the injectivity radius at $p \in M$.

Proposition 3.1. *Let $p, q \in M$ be as in Lemma 2.3. Given $0 < v < 1$, there is an $r \geqslant 0$ so that if $\operatorname{vol}_M > (1 - v)\operatorname{vol}_{S^n}$ and $i(p), i(q) > \pi/2 + r$, then M is homeomorphic to S^n.*

To complete the proof of Theorem B, let us denote the unit tangent sphere at p by $S_p(1)$. For $u \in S_p(1)$ and $r > 0$, set $A(u, r) := \det(\exp_{*|ru})$. Then Bishop's Monotonicity Theorem in [1, §11.10] states that

$$\frac{d}{dr}\left\{\frac{A(u, r)}{\sin^{n-1} r / r^{n-1}}\right\} \leqslant 0.$$

Therefore, $A(u, r) \geqslant \sin^{n-1} r / r^{n-1} - \sin^{n-1} i(p)/i(p)^{n-1}$. Integrating this, we get

$$\operatorname{vol}_M \geqslant \operatorname{vol}_{\hat{B}(i(p))} - \operatorname{vol}_{S^{n-1}} \cdot i(p) \cdot \sin^{n-1} i(p).$$

From the last estimate, we can reduce Theorem B to the situation of Proposition 3.1.

4. Turning now to Main Theorem, the following is an immediate consequence of Bishop's theorem.

Lemma 4.1. *If* $\mathrm{vol}_M > \frac{1}{2}\mathrm{vol}_{S^n}$, *M is simply connected.*

Let $q \in M$. Denote by $S_q(r)$ the sphere of radius r in the tangent space T_qM, and by m its natural measure. Let N_q be the star-shaped domain in T_qM bounded by the tangential cut locus of q. The next observation follows easily from the volume comparison.

Lemma 4.2. *Given any* δ, $r > 0$, *there is an* $v_1 > 0$ *such that if* $\mathrm{vol}_M > (1 - v_1)\mathrm{vol}_{S^n}$, *then* $m(S_q(r) \neg N_q) < \delta$.

Using Fubini's theorem for polar coordinates in $S_q(r)$, we prove

Lemma 4.3. *Given any* η, $r > 0$, *there exists an* v_1, *and if* $\mathrm{vol}_M > (1 - v_1)\mathrm{vol}_{S^n}$, *then any* $u \in S_q(r)$ *can be joined to some* $v \in S_q(r) \cap N_q$ *by a path of length* $< \eta$ *in* $S_q(r)$.

Let us note that from the assumptions in Main Theorem, we can also find a lower bound for the sectional curvature of M in terms of n and κ. Since the norm of the map $\exp_{p*}$ can be estimated from above by such bounds from both sides, we have

Corollary 4.4. *Under the hypotheses of Main Theorem, given any* ε, $r > 0$, *there is an* $v_1 > 0$ *so that if* $\mathrm{vol}_M > (1 - v_1)\mathrm{vol}_{S^n}$, *then any* $x \in \exp(S_q(r))$ *has distance* $d(x, B(r; q)) < \varepsilon/3$.

Recall now that J. Cheeger has obtained a very general lower bound for the injectivity radius from bounds on sectional curvature, volume, and diameter (see [2, Chap. 5]). More recently, E. Heintze and H. Karcher [9] have improved this estimate somewhat. From this, we deduce

Lemma 4.5. *With the same assumptions, given* ε *sufficiently small, there is an* $v_2 > 0$ *such that if* $\mathrm{vol}_M > (1 - v_2)\mathrm{vol}_{S^n}$, *then* $i_M > \varepsilon$.

Thus any ball in M of radius $\leqslant \varepsilon$ is contractible. Now take $p, q \in M$ so that $d(p, q) = d_M$. Set $d_1 := \varepsilon/3$. From Lemma 2.3 and its proof, one sees that for some $v_3 > 0$ and $d_2 := D(v_3) - \varepsilon/3$, whenever $\mathrm{vol}_M > (1 - v_3)\mathrm{vol}_{S^n}$ the balls $B(2\varepsilon/3; p)^-$ and $B(d_2 + \varepsilon/3; q)^-$ cover M. In Corollary 4.4, set $r := d_2 + \varepsilon/3$. We obtain

Lemma 4.6. *We are still in the same situation and* ε, r *are as above. Then there is an* $v > 0$ *such that if* $\mathrm{vol}_M > (1 - v)\mathrm{vol}_{S^n}$, *then* $\exp(S_q(r))$ *is contained in a contractible set* $C := B(\varepsilon; p)$.

For the proof, we simply take $v := \min\{v_1, v_2, v_3\}$.

We can now complete the proof of Main Theorem. Let $D \subset T_qM$ be the closed disc bounded by $S_q(r)$. Let M' be the quotient space M/C, and Π: $M \to M'$ the natural projection. Note that M' can be given the structure of a topological manifold and that Π is a homotopy equivalence. Consider $\Pi \circ \exp_q$: $D \to M'$. Insofar as ∂D is mapped to a point, this map factors through a continuous map h: $S^n \to M'$. Since there is a set in M' which is covered only once, h is seen to have mapping degree 1. Now it is a well-known topological fact that M' and hence M also have the homotopy type of S^n.

5. In this section, we remark that Cheng's Theorem A can also be proved more directly using our geometric techniques. Cheng's original proof relied on the estimates for the eigenvalues of the Laplace operator.

Lemma 5.1. *For any $p \in M$,*

$$\frac{\mathrm{vol}_M - \mathrm{vol}_{B(\pi/2;\, p)}}{\mathrm{vol}_{B(\pi/2;\, p)}} \leqslant 1.$$

This can be obtained from Lemma 2.1 by setting $R = \pi$, $r = \pi/2$ and subtracting 1 from both sides of the inequality. Now suppose that $d_M = \pi$. Choose p, q so that $d(p, q) = \pi$. Then using that $B(\pi/2;\, p)$ and $B(\pi/2;\, q)$ are each contained in the complement of the other, from Lemma 5.1 we obtain

$$\mathrm{vol}_{B(\pi/2;\, q)}/\mathrm{vol}_{B(\pi/2;\, p)} = 1.$$

Subtracting this again from the inequality of Lemma 5.1 gives

Lemma 5.2. *If $d_M = \pi$, then*

$$\mathrm{vol}_M = \mathrm{vol}_{B(\pi/2;\, p)} + \mathrm{vol}_{B(\pi/2;\, q)}.$$

Corollary 5.3. *In the same situation, the two closed balls $B(\pi/2;\, p)^-$ and $B(\pi/2;\, q)^-$ cover M and have a common boundary.*

Thus any geodesic from p to $\partial B(\pi/2;\, p)$ connects to a geodesic to q of length π.

Lemma 5.4. *For M, p as above, $i(p) = \pi$.*

The last assertion follows from the observation that the geodesics emanating from p and entering into $B(\pi/2;\, q)$ all minimize precisely to q, and their initial velocity vectors form an open and closed set in $S_p(1)$.

From Lemma 5.4 and the standard index comparison, it is easy to see that along any of these radial geodesics at p, the vector fields considered in the proof of Myers' theorem are Jacobi fields. This forces the sectional curvature to equal 1 identically in all radial directions from p. But this is enough to construct an isometry from S^n onto M exactly as in [6, §7.3].

6. A closer examination of our proof to Theorem B shows that it suffices to bound the injectivity radii at only two points on M, albeit they need be specially situated with respect to each other. In this context, we mention its relations with the almost-Blaschke manifolds of O. Durumeric. A manifold M is said to be ε-Blaschke at p for some $\varepsilon > 0$, if $i(p) > (1 - \varepsilon)d(p)$; here $d(p) := \sup_{q \in M} d(p, q)$. Recently, Durumeric [4] showed that there is an ε depending only on an arbitrarily given lower bound to the sectional curvature which rather severely restricts the topology of M, ε-Blaschke. However, without the curvature dependence, such manifolds seem to have fairly arbitrary topology even in dimension 2.

Note that the assumption on sectional curvature in our Main Theorem is also only a dependence and not a restriction. It only enters in the last two steps of our proof, though in crucial ways. Since the constant in Theorem B is independent of any sectional curvature, one might hope to eliminate it also from Main Theorem by using some methods different from ours. In fact, a more straight-forward perturbation of Theorem A would be

Problem. Is there a constant $\delta > 0$ depending only on n such that if $d_M > \pi - \delta$, then M is in some sense topologically similar to S^n?

Recent works by K. Grove and K. Shiohama [8] and D. Gromoll and Grove [5] show that if the sectional curvature of M is $\geqslant 1$, its topology is completely determined already if $d_M \geqslant \pi/2$. However, we can find metrics on $S^j \times S^k$ so that $\mathrm{Ric} \equiv j + k - 1$ and the diameter approaches π as $j + k$ goes to ∞. So for the Ricci curvature case, the dependence on n at least seems inevitable.

Addendum. After this work has been completed, we have received oral communications that K. Shiohama has obtained a result apparently similar to ours.

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